ON CLIQUES IN GRAPHS

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ABSTRACT
Sharp bounds are found on the maximal number of sizes of cliques in a graph on n vertices.

Let $G(n)$ be a graph on n vertices. A nonempty set S of vertices of G forms a complete graph if each vertex of S is joined to every other vertex of S. A complete subgraph of G is called a clique if it is maximal i.e., if it is not contained in any any other complete subgraph of G.

Denote by $g(n)$ the maximum number of different sizes of cliques that can occur in a graph of n vertices. Moon and Moser [2] and P. Erdös [1] have obtained surprisingly sharp estimates for $g(n)$. They showed (throughout this paper all logs are to the base 2)

$$n - \log n - H(n) - 0(1) < g(n) - \log n$$

where $H(n)$ is the minimal t such that the t-times iterated logarithm of n is less than 2. Erdös then asked if

$$\lim_{n \to \infty} (g(n) - (n - \log n)) = \infty .$$

In this note we answer this question negatively. We show that for N sufficiently large ($>$ 33000 will do)

$$g(N) \geqq N - \log N - 4.$$

We first give a construction for a specific value of N. Let n sufficiently large ($n \geqq 15$ will do) be given. Define $n_0 = n$, $n_i =$ the minimal integer so that $2^{n_i} + n_i - 2 \geqq n_{i-1}$, $s =$ minimal integer such that $n_s = 2$. Set $A = [\sum_{i=1}^{s} (2^{n_i} + n_i - 1)] + 1$ and $r = [n/2]$. Our points are $y_1, \cdots, y_n$, y^*, disjoint sets C_i, $1 \leqq i \leqq n$ with

Received August 12, 1970 and in revised form November 26, 1970

$|C_i| = 2^{i-1} + 1$, a set C^* with $|C^*| = A$, and a point z. Note that as $A \sim n$, $N \sim 2^n + 3\,n$. Clearly, for n sufficiently large, $r + n_1 + \cdots + n_s + 1 < n$. For convenience we shall also label the points $y_{r+1}, \cdots, y_{r+n_1+\cdots+n_s+1}$ as w_{ij}, $1 \leqq i \leqq s$, $1 \leqq j \leqq n_i$, and $w_{s+1,1}$. We label the points of C^* as v_{ijk} where $1 \leqq i \leqq s$, $1 \leqq j \leqq n_s$, $1 \leqq k \leqq 2^{j-1} + 1$, and the point $v_{s+1,1,1}$. Now for the edges. Make $\{y_1, \cdots, y_n, y^*\}$ complete. Also make $\bigcup_{i=1}^{n} C_i \cup C^*$ complete. If $x \in C_i$ join x to y^* and all $y_j, j \neq i$. If y_i is not a w_{jk} connect it to C^*. Do not connect y^* with any elements of C^*. Connect w_{ij} and $v_{i'j'k}$ if and only if $i = i'$ and $j \neq j'$. Finally, connect z to the w_{ij} and v_{ijk}.

Let $B = 2^n + n - 1 + A$. We claim that this graph contains cliques of all sizes d, $3 \leqq d \leqq B$. For $d = B$, $\bigcup_{i=1}^{n} C_i \cup C^*$ is the desired clique. Now say $d = B - \alpha$, $0 < \alpha < A - 1$. Take the binary expansion $\alpha = \sum_{i=1}^{k} 2^{a_i - 1}$. Then

$$\{y_{a_1}, \cdots, y_{a_k}\} \cup C^* \cup \bigcup_{j \neq a_i} C_j$$

is a clique with d elements. (Since $r > \log A$ no y_{a_i} is a w and therefore the y_{a_i} are all connected to C^*. Completeness easily follows. As y_j and C_j are not connected no other y_j or C_j can be added. y^* cannot be added as it is not connected to C^*. z cannot be added as either y_1 or C_1 is in the set, neither of which are connected to z.) Now say $d = B - (A-1) - \alpha$, $0 \leqq \alpha \leqq 2^n - 1$. Again take the binary expansion $\alpha = \sum_{i=1}^{k} 2^{a_i - 1}$. Then

$$\{y^*, y_{a_1}, \cdots, y_{a_k}\} \cup \bigcup_{j \neq a_i} C_j$$

is a clique with d elements. Now say $3 < d \leqq n$. By the construction of the n_i we find i, $n_i < d - 1 < 2^{n_i} + n_i - 1$. We find the binary expansion $2^{n_i} + n_i - 1 - (d-1) = \sum_{j=1} 2^{b_j}$. Then

$$\{z, w_{ib_1}, \cdots, w_{ib_t}\} \cup \{v_{ijk} : j \neq b_q \text{ for } 1 \leqq q \leqq t\}$$

is a clique with d elements. (Here the completeness is straightforward. As $d - 1$ satisfies strict inequalities there is at least one w and one v in the set. If a point could be added then, since it would be connected to z, it would be of the form $w_{i'j'}$ or $v_{i'j'k'}$. As some w is in the set, any v to be added must have $i = i'$. As some v is in the set, any w to be added must have $i = i'$. If w_{ij} is not in the set then v_{ijk} is for $k = 1$ and so w_{ij} cannot be added. Similarly, no v_{ijk} can be added.)

Finally, $\{z, w_{s+1,1}, v_{s+1,1,1}\}$ is a 3-clique.

Let $f(n)$ be the number of elements in this graph. We have shown that for

$N = f(n)$, $g(N) \geqq N - \{\log N\} - 3$. Now say that $f(n) < N < 2^n + 2^{r-2}$. Add $N - f(n)$ points to C^*. These points are connected to each other and all points except y^* and z. Set $A =$ the new $|C^*|$, $B = 2^n + n - 1 + A$. This graph has cliques of all sizes d, $3 \leqq d \leqq B$, the proof reading as before. So, for these N, $g(N) \geqq N - \{\log N\} - 3$.

Now say

$$2^n + 2^{r-2} \leqq N < f(n+1) < f(n) + 2^n + 5n.$$

Set $n_0 = n + 1$, adjust n_i accordingly, and construct the graph as before. It will have $f_1(n)$ points where $0 \leqq f_1(n) - f(n) < n$ (in fact, is very small). Add $10n$ points to C^*, connected to each other and all points except y^* and z. Add a point y_{n+1} and a set C_{n+1} with $N - f_1(n) - 10n - 1 < 2^n$ points. Extend the definitions of edges for y_i and C_i to $n + 1$. Set $A = |C^*|$, $B = |\bigcup C_i \cup C^*|$. This graph has cliques of all sizes d, $3 \leqq d \leqq B$. For $d = B - \alpha$, $0 \leqq \alpha < A - 1$ the proof is the same. For $d = B - (A-1) - \alpha$, $0 \leqq \alpha \leqq 2^n - 1$ it is the same. For $d = B$ $(A-1) - |C_{n+1}| - \alpha$, $0 \leqq \alpha \leqq 2^n - 1$ we take

$$\alpha = \sum_{i=1}^{k} 2^{a_i - 1}$$

and

$$\{y^*, y_{a_1}, \cdots, y_{a_k}, y_{n+1}\} \cup \bigcup_{\substack{j \neq a_i \\ j \neq n+1}} C_j$$

is the clique with d elements. For $3 \leqq d \leqq n+1$ the proof is as before. Thus $g(N) \geqq N - \{\log N\} - 4$.

REFERENCES

1. P. Erdös, *On cliques in graphs*, Israel J. Math. **4** (1966), 233–234.
2. J. W. Moon, and L. Moser, *On cliques in graphs*, Israel J. Math. **3** (1965), 23–28.

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